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The development of time domain electromagnetic solvers for nonorthogonal grids is an area of current research interest, stemming from the need to simulate complex geometries in a wide variety of applications. A notable example is the discrete surface integral (DSI) algorithm [1], which solves the Maxwell curl equations in the time domain using a 3d, unstructured, mixed-polyhedral grid. Although this method is an extension of the time proven Yee [2] algorithm, little is known about the numerical properties of the method when discretized on these more general grids. Dispersion relations for the DSI algorithm can be derived using 2d idealized grids, such as the skewed mesh analysis done by Ray and Rambo [3] for both triangles and quadrilaterals. The present work applies the same techniques used for the skewed mesh analysis to another idealized, but nonorthogonal, 2d grid.

The mesh examined here is the regular, periodic "chevron" (herring-bone, cork-screw, saw-tooth) mesh (see Fig. 1). This mesh is simple enough to be easily analyzed while still stressing an algorithm's ability to handle nonorthogonal meshes. We consider the source-free Maxwell's equations in vacuum for waves with electric field **E** polarized in the *x*-*y* plane and magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$. The plane is periodically tiled with identical chevron unit cells, shown in Fig. 1, labeled by indices (l,m); the grid is characterized by Δx, Δy and the chevron angle ϑ. The locations and vector orientations of the field variables are also detailed in Fig. 1. The dual grid, which happens to be an orthogonal grid of rectangles, is shown by dashed lines.

The dispersion analysis proceeds by applying the DSI algorithm to determine the update equations on the particular grid shown in Fig. 1. Due to the orthogonality of the dual mesh to the vertical primary edges, the updates for E_2 and $\tilde{E}_2$ are simple differences of adjacent magnetic fields. The magnetic fields are determined by side length weighted sums of the surrounding tangential electric fields. Only the components E_1 and $\tilde{E}_1$ are affected by the nonorthogonality of the mesh. However, applying the DSI prescription leads to updates for these components in a straight forward manner. Once the update

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[1] Niel K. Madsen, "Divergence Preserving Discrete Surface Integral Methods for Maxwell's Curl Equations Using Non-Orthogonal Unstructured Grids," UCRL-JC-109787 (February 1992), and submitted to J. Comput. Phys.
[2] K. S. Yee, "Numerical Solution of Initial Boundary Value Problems in Isotropic Media," IEEE Trans. Antennas Propagat. **AP-14**, 302 (1966).
[3] Scott L. Ray, "Numerical Dispersion and Stability Characteristics of Time Domain Methods on Nonorthogonal Meshes," IEEE Trans. Antennas Propagat. **41**, 233 (1993); P. W. Rambo, "Dispersion and Stability of Two Electromagnetic Solvers for Triangular Meshes," Proceedings of the 14th Int. Conf. on the Numerical Simulation of Plasmas, (Annapolis MD, 1991); S. L. Ray and P. W. Rambo, "A Comparative Study of the Accuracy of Various Time Domain Nonorthogonal Mesh Methods," Proceedings of the IEEE-APS Int. Symposium (Chicago IL 1992).

equations are known, all field components are expressed as Fourier modes varying as $\exp(-i\omega t_n + i\mathbf{k}\cdot\mathbf{X}_{l,m})$, where $t_n = n\Delta t$, $\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$ and $\mathbf{X}_{l,m} = l(2\Delta x)\hat{\mathbf{x}} + m\Delta y\hat{\mathbf{y}}$ identifies each unit cell in the periodic grid. This system of linear equations for the six field amplitudes may then be solved to obtain the dispersion relation for the chevron grid,

$$\left\{-4\sin^2\frac{\omega\Delta t}{2} + \frac{(c\Delta t)^2}{\Delta x\,\Delta y}\left(2\frac{\Delta y}{\Delta x} + 4\frac{\Delta x}{\Delta y}\sin^2\frac{k_y\Delta y}{2}\right)\right\}^2$$

$$= \left\{\frac{(c\Delta t)^2}{\Delta x\,\Delta y}\right\}^2 \left\{\left(2\frac{\Delta y}{\Delta x}\cos k_x\Delta x\right)^2 - \left(\tan\vartheta\,\sin k_y\Delta y\,\sin k_x\Delta x\right)^2\right\}. \quad (1)$$

Expanding Eq. (1) for small wavenumber confirms convergence with second order accuracy. For the case of $\vartheta=0$, the usual orthogonal grid dispersion relation is obtained as expected; this is also true for $\vartheta\neq 0$ if either $k_x=0$ or $k_y=0$.

With the mesh perturbed ($\vartheta\neq 0$), the right hand side of Eq. (1) may be negative leading to complex frequencies, $\omega = \omega_r + i\gamma$ (ω_r and γ both real). This condition for instability may be expressed as,

$$|\tan\vartheta| \;>\; \left|2\frac{\Delta y}{\Delta x}\,\mathrm{ctn}(k_x\Delta x)\,\csc(k_y\Delta y)\right|, \quad (2)$$

which is satisfied for any nonzero ϑ when $k_x\Delta x=\pi/2$ (and $k_y\Delta y\neq 0$), and is independent of the time step. Expanding the dispersion relation for small time step with this condition satisfied, we find nonzero growth rate ($\gamma>0$) for vanishing time step. This is very different from the usual Courant limit, since it implies that no stable time step exists. As a simple concrete example, consider a perturbed square mesh, $\Delta x=\Delta y\equiv\Delta$, with the choice $k_x\Delta x=k_y\Delta y=\pi/2$. The growth rate and real part of the frequency in the limit of small time step are found to be,

$$\gamma = \frac{c}{4\Delta}\tan\vartheta \quad , \quad \omega_r = \frac{2c}{\Delta}. \quad (3)$$

(This value for the growth rate is within 4% of the maximum growth rate, which occurs at $k_y\Delta y=1.30$).

Numerical tests on chevron grids confirm the presence of this instability. Periodic 2d simulations verify the dispersion relation in detail. Tests performed in 3d, with the simulation region surrounded by a conducting boundary, also show the growth rate to be independent of Δt, proportional to $\tan\vartheta$, and inversely proportional to the mesh scale length as predicted. Realistic grids with isolated regions of chevron zones have also been observed to be unstable.

When applied to the chevron grid investigated here, the DSI algorithm supports electromagnetic oscillations which are unstable at any time step. Application of the Modified Finite Volume (MFV) algorithm [4] to this mesh results in identical update equations, and hence identical dispersion and stability properties. We suspect that other nonorthogonal grid methods currently in use which are neither dissipative nor provably-stable [5] may encounter similar difficulties. The impact of this instability on real problems is difficult to quantify. Any non-trivial mesh will likely contain some chevron zones, although perhaps only in limited spatial regions. Absorbing boundaries can serve to keep unstable modes at low, but not necessarily negligible levels. In some instances, mesh refinement may make the problem more severe, since the growth rate is inversely proportional to mesh size. For many problems, the growth rate may be small enough to not affect the solution, however, more work is required to quantify the seriousness of this instability.

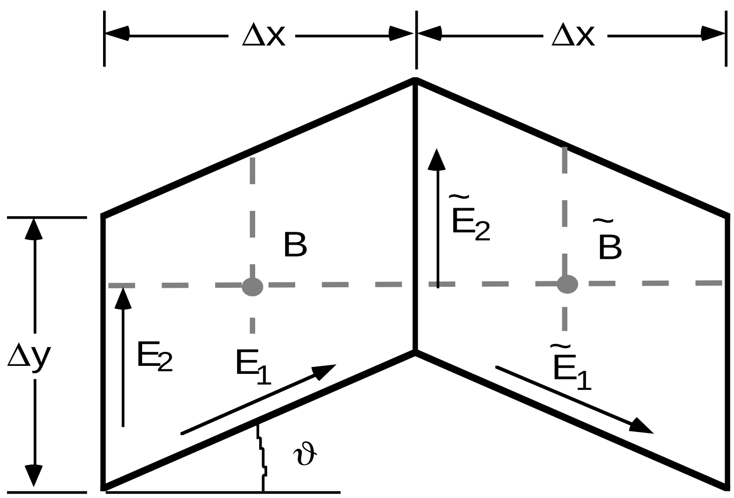

Figure 1

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[4] N. K. Madsen and R. W. Ziolkowski, "Numerical Solution of Maxwell's Equations in the Time Domain Using Irregular Nonorthogonal Grids," Wave Motion **10**, 583 (1988).

[5] A. B. Langdon, "On Enforcing Conservation Laws in Electromagnetic Particle-in-Cell Codes," Proceedings of the 14th Int. Conf. on the Numerical Simulation of Plasmas (Annapolis MD, 1991).

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